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ON THE HYBRID STRESS FINITE ELEMENT MODEL FOR INCREMENTAL ANALYSIS OF LARGE DEFLECTION PROBLEMS

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Abstract—An extension of the hybrid stress finite element model, originally suggested for linear small displacement problems by Pian, to the large deflection problem is presented. An incremental approach and the concepts of initial stress are employed. A procedure to check the equilibrium in the reference state, prior to the addition of a further load increment, is included. An example and a discussion of the results are presented.

INTRODUCTION

THE FORMULATION of the finite element methods as being based on rigorous variational principles in solid mechanics and their modifications is comparatively of recent origin, see for example Pian and Tong [1], Atluri and Pian [2] and Oden [3] for a more comprehensive bibliography. In the commonly used displacement models, it is widely recognized that for monotonic convergence of the finite element solution, the assumed element displacement functions should satisfy the interelement boundary displacement compatibility, which is by no means a simple task in several problems of practical interest such as plates and general shells. In 1964, Pian [4] suggested a method, which was later discussed more rigorously by Pian and Tong [5], in which the interelement boundary displacement compatibility can be satisfied rather easily. In this approach, one assumes an equilibrium stress field in the interior of the element, and a displacement field at the boundary of the element, which inherently satisfies the interelement compatibility condition. Pian's approach has since been employed in plate bending problems [6–8], St. Venant torsion problems [9], shell problems [10], analysis of multi-layered plates and shells [11] where transverse shear effect is important, and in the analysis of stress states around cracks [12], all of which clearly demonstrated the versatility of the hybrid stress model for deriving stiffness matrices of elements of arbitrary geometry. All the above mentioned solutions were limited to small-deflection linear elastic problems.

Using the above approach, some satisfactory results were obtained for buckling problems by Lundgren [13] and for incremental analysis of large deflection problems by Pirotin [14]. As discussed by Pian [15], these methods cannot be considered as being based on the modified complementary energy principle which is the basis of the hybrid stress model, and hence cannot be considered as consistent hybrid stress finite element methods.†

† In a recent paper, Pian [16], however, has shown that the above methods can be justified as being based on the modified Reissner–Hellinger variational principle, even though they are not the hybrid stress models as meant here.

Incremental formulations of finite displacement large-strain problems, using a compatible displacement finite element model, have also been presented by Hofmeister *et al.* [17], Pian and Tong [18], Stricklin *et al.* [19] and Washizu [20]. A significant conclusion in Refs. [17, 19] was that a "check" on the equilibrium in the current reference state prior to the addition of a further load increment was indispensible for obtaining meaningful computational results in an incremental formulation of large strain, nonlinear problems.

Since, as discussed by Pian [15], the hybrid assumed stress finite element model has proved to be a more versatile structural tool, it is the purpose of the present note to present a consistent variational formulation of the hybrid stress model for the incremental analysis of large deflection problems. The concept of initial stress is employed, wherein, during any given step in which the displacements, stresses and external loads undergo increments, the state at the beginning of the step is considered as one of initial stress. A "check" on the equilibrium in the state prior to the addition of further load increment, analogous to the one suggested by Hofmeister *et al.* [17], is included.

BASIC FORMULATION

For clarity in presentation, the ideas will be developed within the framework of three-dimensional elasticity theory. Some of the initial development of the theory follows closely that of Washizu [21], and Truesdell and Toupin [22] and is repeated here for the sake of completeness.

Each material particle in the three-dimensional continuum in the original reference configuration C_1 is identified, in general, by three curvilinear coordinates ξ^α ($\alpha = 1, 2, 3$). The numerical values of ξ^α which define a particle in C_1 define the same particle in every subsequent configuration (also referred to as convected coordinates). To describe the motion of the body relative to C_1, a fixed rectangular cartesian coordinate system x_α ($\alpha = 1, 2, 3$) in three-dimensional space is also established. Thus, a continuous one-to-one motion of the particle, as the continuum deforms, is defined by relations,

$$x_\alpha = x_\alpha(\xi^\beta, t) \tag{1}$$

such that

$$\det\left|\frac{\partial x_\alpha}{\partial \xi^\beta}\right| > 0. \tag{1a}$$

In general, we define C_N to be the configuration of the body before the addition of the nth increment of load; whereas, C_{N+1} is the configuration of the body after the addition of the nth load-increment. In configuration C_N, the states of stress, strain and deformation are presumed to be known. During the process of the nth load-increment, the configuration C_N is treated to be in a state of "initial stress". Incremental displacements due to the addition of the nth load-increment are measured from C_N. In the following we treat, as a generic case, the movement of the continuum from the reference state C_N to further deformed state C_{N+1} through small but finite increments in stresses, displacements and external loads.

The position vector of a particle in C_N is denoted by $\mathbf{r}$ and that of the same particle in C_{N+1} is denoted by $\mathbf{R}$. If x_i are the cartesian coordinates of the point in C_N and $\mathbf{e}_i$ are cartesian bases, it follows

$$\mathbf{r} = x_i \mathbf{e}_i \tag{2}$$

The covariant base vectors tangent to ξ^i lines in configuration C_N are then given by,

$$\mathbf{g}_\alpha = \frac{\partial \mathbf{r}}{\partial \xi^\alpha} = \frac{\partial x_i}{\partial \xi^\alpha}\mathbf{e}_i \tag{3}$$

the covariant and contravariant metric tensors, and contravariant base vectors in C_N are given by,

$$g_{\alpha\beta} = \mathbf{g}_\alpha \cdot \mathbf{g}_\beta; \qquad [g^{\alpha\beta}] = [g_{\alpha\beta}]^{-1}; \qquad \mathbf{g}^\alpha = g^{\alpha\beta}\mathbf{g}_\beta \tag{4}$$

The vector field of incremental displacement from C_N to C_{N+1}, is measured in the basis system of C_N as

$$\Delta\,\mathbf{u} = \Delta u_\alpha \mathbf{g}^\alpha = \Delta u^\alpha \mathbf{g}_\alpha \tag{5}$$

which gives the usual representation of Green's strain tensor, in C_N as

$$\Delta\varepsilon_{\lambda\mu} = \tfrac{1}{2}[\Delta u_{\lambda,\mu} + \Delta u_{\mu,\lambda} + \Delta u^{\nu}_{,\lambda}\Delta u_{\nu,\mu}] \tag{6}$$

where a comma denotes a covariant differentiation with respect to coordinates ξ^i in C_N, using the metric tensors g_{ij}, g^{ij} of C_N. For later use we note that the geometry of C_{N+1} is characterized by the basis vectors and metric tensors,

$$\mathbf{G}_\alpha = \frac{\partial \mathbf{R}}{\partial \xi^\alpha} = \frac{\partial(\mathbf{r}+\mathbf{u})}{\partial \xi^\alpha} = (\delta^\beta_\alpha + u^\beta_\alpha)\mathbf{g}_\beta; \qquad G_{\alpha\beta} = g_{\alpha\beta} + 2\Delta\varepsilon_{\alpha\beta}. \tag{7}$$

In the reference state C_N, which is presumed to be known, let the initial Piola–Kirchoff stresses be represented by the symmetric tensor $\sigma^{o\lambda\mu}$, measured per unit area in C_N. Let the initial body forces and surface tractions measured per unit volume and unit area respectively, in the current reference state, be given, respectively, by

$$\bar{F}^o = \bar{F}^{o\lambda}g_\lambda; \qquad \bar{T}^o = \bar{T}^{o\lambda}g_\lambda \tag{8}$$

where a bar (—) denotes a prescribed quantity. One can then prescribe additional body forces $\Delta\bar{F}^\lambda$, additional surface tractions $\Delta\bar{T}^\lambda$ on a portion S_1 of the surface of the body, and additional displacements $\Delta\bar{u}^\lambda$ on a portion S_2 of the surface of the body; where the displacements Δu^λ are measured from C_N, in the basis vector system $\bar{g}_\lambda$ of C_N. Let the corresponding increment in the Piola–Kirchoff stresses measured per unit area in C_N be represented by the symmetric tensor $\Delta\sigma^{\lambda\mu}$. The principle of virtual work then states,

$$\int_V [(\sigma^{o\lambda\mu} + \Delta\sigma^{\lambda\mu})\delta\Delta e_{\lambda\mu} - (\bar{F}^{o\lambda} + \Delta\bar{F}^\lambda)\delta\Delta u_\lambda)\,\mathrm{d}V - \int_{S_1} (\bar{T}^{o\lambda} + \Delta\bar{T}^\lambda)\delta\Delta u_\lambda\,\mathrm{d}S = 0 \tag{9}$$

where $\Delta e_{\lambda\mu}$ is given by equation (6). In equation (9) the volume V, and the surfaces S_1 and S_2 refer to the known current reference state C_N, and a "δ" denotes variations. We note that $\delta\Delta u_\lambda$ vanishes on S_2. Equation (9) can be written as,

$$\int_V \left[\Delta\sigma^{\lambda\mu}\delta\Delta e_{\lambda\mu} + \frac{\sigma^{o\lambda\mu}}{2}\delta(\Delta u_{\nu,\lambda}\Delta u^{\nu}_{,\mu}) - \Delta F^\lambda\delta\Delta u_\lambda\right]\mathrm{d}V - \int_{S_1} \Delta\bar{T}^\lambda\delta\Delta u_\lambda\,\mathrm{d}S$$
$$= \int_V \left[-\frac{\sigma^{o\lambda\mu}}{2}(\delta\Delta u_{\lambda,\mu} + \delta\Delta u_{\mu,\lambda}) + F^{o\lambda}\delta\Delta u_\lambda\right]\mathrm{d}V + \int_{S_1} T^{o\lambda}\delta\Delta u_\lambda\,\mathrm{d}S. \tag{10}$$

If it is now assumed that the known initial stress state ($\sigma^{o\lambda\mu}$; $\bar{F}^{o\lambda}$; $\bar{T}^{o\lambda}$) in C_N is in equilibrium prior to the addition of the incremental loads for step N, then the right hand side of equation (10) can be shown to be identically equal to zero. However, as pointed out by Hofmeister *et al.* [17], due to the numerical incremental solution technique for solving a large strain problem, the initial stress state in C_N *may* not be in equilibrium. Following Hofmeister *et al.* [17], it is shown later that it is possible to derive an equilibrium error check if the right hand side terms in equation (10) are retained.

Assuming that the elastic stress–strain relations are of the type

$$\Delta\sigma^{\lambda\mu} = \Delta\sigma^{\lambda\mu}(\sigma^{o\alpha\beta}, \Delta e_{\alpha\beta}) \tag{11}$$

or

$$\Delta e_{\lambda\mu} = \Delta e_{\lambda\mu}(\sigma^{o\alpha\beta}, \Delta\sigma^{\alpha\beta}) \tag{11a}$$

one can define an elastic strain energy function

$$\mathrm{d}A = \Delta\sigma^{\lambda\mu}\,\mathrm{d}\Delta e_{\lambda\mu}. \tag{12}$$

Using equation (12) equation (10) may be written as,

$$\delta\left\{\int_V [A(\sigma^{o\alpha\beta}, \Delta u^{\alpha}) + \tfrac{1}{2}\sigma^{o\lambda\mu}\Delta u_{\nu,\mu}\Delta u^{\nu}_{,\mu} - \Delta\bar{F}^{\lambda}\Delta u_{\lambda}]\,\mathrm{d}V - \int_{S_1}\Delta\bar{T}^{\lambda}\Delta u_{\lambda}\,\mathrm{d}S\right\}$$

$$= \int_V (-\sigma^{o\lambda\mu}\delta\Delta u_{\lambda,\mu} + F^{o\lambda}\delta\Delta u_{\lambda})\,\mathrm{d}V + \int_{S_1}\bar{T}^{o\lambda}\delta\Delta u_{\lambda}\,\mathrm{d}S. \tag{13}$$

It must be stressed again that the right hand side in equation (13) is a correction term to "check" that the initial stresses in C_N satisfy the equilibrium equations and boundary conditions. Thus, theoretically, if the reference state C_N is one of equilibrium, in which case the right hand side of equation (13) vanishes, the equation (13) can be shown to yield the equilibrium equations, for the Piola–Kirchoff incremental stresses (due to nth loading increment) referred to the current known reference state C_N, as follows:

$$\Delta\sigma^{\lambda\mu}_{,\mu} + [(\sigma^{o\nu\mu} + \Delta\sigma^{\nu\mu})\Delta u^{\lambda}_{,\mu}]_{,\nu} + \Delta\bar{F}^{\lambda} = 0 \tag{14}$$

and

$$\Delta\bar{T}^{\lambda} = \Delta\sigma^{\lambda\mu}n_{\mu} + (\sigma^{o\nu\mu} + \Delta\sigma^{\nu\mu})\Delta u^{\lambda}_{,\mu}n_{\nu} \quad \text{on } S_1. \tag{15}$$

The principle of virtual work as given by equation (13) can now be generalized through the usual methods, into a counterpart of Hu-Washizu variational principle in linear elasticity [21]. That is, we add to the functional in equation (13) the constraint conditions

$$\Delta e_{\lambda\mu} = \tfrac{1}{2}(\Delta u_{\lambda,\mu} + \Delta u_{\mu,\lambda} + \Delta u^{\nu}_{,\mu}\Delta u_{\nu,\lambda}) \tag{16}$$

and

$$\Delta u_{\lambda} = \Delta\bar{u}_{\lambda} \quad \text{on } S_2.$$

Then considering the Lagrangian multipliers corresponding to constraint conditions (16 and 16a) as $\Delta\sigma^{\lambda\mu}$ and ΔT^{λ}, respectively, we can formulate the generalized functional,

$$\pi_g = \pi_g(\Delta e_{\lambda\mu}; \Delta u_{\lambda}; \Delta\sigma^{\lambda\mu}; \Delta T^{\lambda}) \tag{17}$$

where

$$\pi_g = \int_V \{A(\Delta e_{\lambda\mu}, \sigma^{o\lambda\mu}) + \tfrac{1}{2}\sigma^{o\lambda\mu}\Delta u^{\nu}_{,\lambda}\Delta u_{\nu,\mu} - \Delta \bar{F}^{\lambda}\Delta u_{\lambda}$$

$$- \Delta\sigma^{\lambda\mu}[\Delta e_{\lambda\mu} - \tfrac{1}{2}(\Delta u_{\lambda,\mu} + \Delta u_{\mu,\lambda} + \Delta u^{\nu}_{,\mu}\Delta u_{\nu,\lambda})]\}\, \mathrm{d}V$$

$$- \int_{S_1} \Delta \bar{T}^{\lambda}\Delta u_{\lambda}\, \mathrm{d}S - \int_{S_2} \Delta T^{\lambda}(\Delta u_{\lambda} - \Delta \bar{u}_{\lambda})\, \mathrm{d}S - \varepsilon^* \tag{18}$$

where

$$\varepsilon^* = \int_V (-\sigma^{o\lambda\mu}\Delta u_{\lambda,\mu} + F^{o\lambda}\Delta u_{\lambda})\, \mathrm{d}V + \int_{S_1} \bar{T}^{o\lambda}\Delta u_{\lambda}\, \mathrm{d}S \tag{19}$$

ε^* is the correction term to "check" the equilibrium of initial stress state in the reference state C_N. Noting that the variation of ε^* with respect to Δu_{λ} is zero if the reference state equilibrium is theoretically satisfied, one can show that the Euler equations corresponding to $\delta\pi_g = 0$ are, (a) the equilibrium equations, equation (14); (b) the strain-displacement relations, equation (6); (c) the stress–strain relations, equation (12); (d) the displacement boundary conditions $\Delta u_{\lambda} = \Delta \bar{u}_{\lambda}$ on S_2; and (e) the stress boundary conditions, equation (15).

Suppose now that in equation (18) one assumes that the incremental equilibrium equations, equation (14), the traction conditions, equation (15), and the stress–strain relations (12) are satisfied *a priori*. Noting that assuming equation (12) *a priori* is equivalent to assuming the existence of a potential B such that

$$B = \Delta\sigma^{\lambda\mu}\Delta e_{\lambda\mu} - A \tag{20}$$

and using equations (14, 15 and 20), one can reduce π_g to π_c, where

$$\pi_c = -\int_V B(\Delta\sigma^{\lambda\mu}; \sigma^{o\lambda\mu})\, \mathrm{d}V + \int_{S_2} \Delta T^{\lambda}\Delta \bar{u}_{\lambda}\, \mathrm{d}S - \varepsilon^* \tag{21}$$

obviously, when π_c is varied with respect to $\Delta\sigma^{\lambda\mu}$ *only* and noting that the variations $\delta\Delta\sigma^{\lambda\mu}$ satisfy the *a priori* conditions

$$\delta\Delta^{\lambda\mu}_{,\mu} + \delta[(\sigma^{o\nu\mu} + \Delta\sigma^{\nu\mu})\Delta u^{\lambda}_{,\mu}]_{,\nu} = 0 \tag{22}$$

and

$$\delta\Delta\sigma^{\lambda\mu}n_{\mu} + \delta[\sigma^{o\nu\mu} + \Delta\sigma^{\nu\mu}]\Delta u^{\lambda}_{,\mu}n_{\nu} = 0 \tag{23}$$

it can be shown that the Euler equations corresponding to the variation of π_c with respect to $\Delta\sigma^{\lambda\mu}$, are

$$\Delta e_{\lambda\mu} = \tfrac{1}{2}(\Delta u_{\lambda\mu} + \Delta u_{\mu,\lambda} + \Delta u^{\nu}_{,\lambda}\Delta u_{\nu,}{}^{\lambda}) \tag{24}$$

and

$$\Delta u_{\lambda} = \Delta \bar{u}_{\lambda} \quad \text{on } S_2. \tag{25}$$

Even though the functional ε^* in equation (21) is a constant with respect to variations in $\Delta\sigma^{\lambda\mu}$ and hence doesn't contribute any terms in the variational equation $\delta\pi_c = 0$ when the variations are with respect to $\Delta\sigma^{\lambda\mu}$, it is shown later that within the framework of the hybrid stress finite element model, an equilibrium check on the initial stresses can be performed by retaining ε^* in equation (21). If the condition $\Delta T^\lambda = \Delta \bar{T}^\lambda$ on S_1, is not satisfied *a priori*, then we can introduce it as a subsidiary condition and consider

$$\pi_c^* = -\int_V B(\Delta\sigma^{\lambda\mu};\sigma^{o\lambda\mu})\,\mathrm{d}V + \int_{S_2} \Delta T^\lambda\,\Delta\bar{u}_\lambda\,\mathrm{d}S + \int_{S_1} (\Delta\bar{T}^\lambda - \Delta T^\lambda)\,\Delta u_\lambda\,\mathrm{d}S - \varepsilon^*. \tag{26}$$

Next, certain simplifications can be made, in order to facilitate the above developments for numerical calculations. One can assume that each increment is such that the incremental displacements Δu_λ are of order $O(\varepsilon)$; whereas the initial stresses are of order $O(1)$. Thus,

$$\Delta u_\lambda \sim O(\varepsilon) \tag{27}$$

$$\sigma^{o\lambda\mu} \sim O(1). \tag{28}$$

Then, the incremental strain-displacement relations are

$$\Delta e_{\lambda\mu} = \tfrac{1}{2}(\Delta u_{\lambda,\mu} + \Delta u_{\mu,\lambda}) + O(\varepsilon^2). \tag{29}$$

Likewise, for elastic materials

$$\Delta\sigma^{\lambda\mu} \sim O(\varepsilon). \tag{30}$$

Using equations (27 and 30), the equilibrium equation (14) can be simplified as,

$$\Delta\sigma^{\lambda\mu}_{,\mu} + [\sigma^{o\nu\mu}\Delta u^\lambda_{,\mu}]_{,\nu} + \Delta\bar{F}^\lambda = 0 \tag{31}$$

and the traction vector ΔT^λ can be simplified as,

$$\Delta T^\lambda = \Delta\sigma^{\lambda\mu} n_\mu + \sigma^{o\nu\mu}\Delta u^\lambda_{,\mu} n_\nu. \tag{32}$$

From equations (29, 31 and 32), it is clear that if in the functional

$$\pi_c^* = -\int_V B(\Delta\sigma^{\lambda\mu},\sigma^{o\lambda\mu})\,\mathrm{d}V + \int_{S_2} \Delta T^\lambda\,\Delta\bar{u}_\lambda\,\mathrm{d}S + \int_{S_1} (\Delta\bar{T}^\lambda - \Delta T^\lambda)\,\Delta u_\lambda\,\mathrm{d}S - \varepsilon^* \tag{33}$$

we assume *a priori*, the linearized equilibrium equation in terms of incremental and initial Piola–Kirchoff stresses, taken per unit area in C_N, as

$$\Delta\sigma^{\lambda\mu}_{,\mu} + [\sigma^{o\nu\mu}\,\Delta u^\lambda_{,\mu}]_{,\nu} + \Delta\bar{F}^\lambda = 0 \tag{34}$$

where a comma in equation (34) refers to covariant differentiation with respect to ξ^i lines in C_N (using metric tensors $g_{\alpha\beta}$ and $g^{\alpha\beta}$ in C_N), then the variational equation

$$\delta_{\Delta\sigma^{\lambda\mu}}\pi_c^* = 0 \tag{35}$$

leads to the Euler equations

$$\Delta e_{\lambda\mu} = \tfrac{1}{2}(\Delta u_{\lambda,\mu} + \Delta u_{\mu,\lambda}) \tag{36}$$

and

$$\Delta u^\lambda = \Delta\bar{u}^\lambda \quad \text{on } S_2. \tag{37}$$

Thus, any incremental stress field that satisfies the incremental equilibrium equation (34) exactly also leads to compatible displacement fields if the variational equation (35) is satisfied. Thus the above variational principle is consistent in contrast to that by Pirotin

[13], in whose formulation the Hellinger–Reissner's principle is used and the incremental stresses are made to satisfy only the equation $\Delta\sigma^{\lambda\mu}_{,\mu}+\Delta\bar{F}^{\lambda} = 0$, instead of equation (34).

Following Pian and Tong [1], one can modify the functional in equation (33) for the hybrid stress formulation of the finite element model as

$$\pi_c^* = \sum_{n=1}^{M} \left\{ -\int_{V_n} B(\Delta\sigma^{\lambda\mu}, \sigma^{o\lambda\mu})\, dV + \int_{\partial V_n} \Delta T^{\lambda}\, \Delta u_{\lambda L}\, dS - \varepsilon^* \right\} \tag{38}$$

where M is the number of elements, V_n the volume of the nth element, ∂V_n the interelement boundary of the nth element. In the above, $\Delta u_{\lambda L}$, which are physically the interelement boundary displacements, take on the meaning of Lagrangian multipliers that can be used to satisfy the interelement traction continuity requirement on the average. These interelement boundary displacements $u_{\lambda L}$ are prescribed such that they inherently satisfy the interelement displacement compatibility condition.

Thus, in the construction of the finite element model, one assumes, (a) an equilibrating stress field within each element, and (b) a set of element boundary displacements in terms of a finite number of nodal values such that interelement displacement compatibility is inherently satisfied. The incremental stress field within each element satisfies the equation

$$\Delta\sigma^{\lambda\mu}_{,\mu} + [\sigma^{o\nu\mu}\, \Delta u^{\lambda}_{,\mu}]_{,\nu} + \Delta\bar{F}^{\lambda} = 0 \tag{39}$$

or

$$\Delta\sigma^{\lambda\mu}_{,\mu} = -\Delta\bar{F}^{\lambda} - (\sigma^{o\nu\mu}\, \Delta u^{\lambda}_{,\mu})_{,\nu}. \tag{40}$$

Thus, we can assume, in each element

$$\{\Delta\sigma\} = [H]\{B\} + \{\Delta\sigma_p\} + [A]\{\Delta q\} \tag{41}$$

where $[H]$ is the matrix of homogeneous solution, and $\Delta\sigma_p$ is any particular solution corresponding to an increment in external loads. The meaning of the last term is explained below.

The coefficient matrix A results from the last forcing term in equation (40), where the initial stress $\sigma^{o\lambda\mu}$ is known. Also, in the hybrid stress method we assume compatible interelement boundary displacements as

$$\{\Delta u_{\lambda L}\} = [L_B]\{\Delta q\} \tag{42}$$

where Δq is the vector of generalized nodal displacements, and L_B is a matrix of interpolation polynomials in terms of the boundary coordinates. From these boundary data one can interpolate for the interior displacements† as

$$\{\Delta u\} = [L]\{\Delta q\}. \tag{43}$$

Using equation (43) the necessary interior displacement gradients may easily be obtained. From these one can construct the matrix A such that,

$$[A]\{\Delta q\} \equiv [(\sigma^{o\nu\mu}\, \Delta u^{\lambda}_{,\mu})_{,\nu}]. \tag{44}$$

Similarly, from (32) one obtains for the element boundary tractions

$$\{\Delta T\} = [R][\beta] + [\Delta T_p] + [M][\Delta q] \tag{45}$$

† As discussed by Pian [16], the interpolation functions in L need not satisfy the interelement boundary compatibility, i.e. in principle and $[L]$ and $[L_B]$ can be independent.

Finally, from a stress–strain law of the type shown by equation (11a) one may construct the following strain–stress relation†

$$\Delta e_{\alpha\beta} = [E^{o}_{\alpha\beta\varepsilon\rho} + 2E^{1}_{\alpha\beta\gamma\delta\varepsilon\rho}\sigma^{o\gamma\delta}]\Delta\sigma^{\varepsilon\rho} + H.0.T. \tag{46}$$

or

$$\Delta e_{\alpha\beta} = E_{\alpha\beta\gamma\delta}(\sigma^{o\nu\mu})\,\Delta\sigma^{\gamma\delta}. \tag{47}$$

We note that the compliance depends on the initial stress state when material nonlinearities are included.

Now one is in the position to facilitate equation (38) for numerical computation. Using equation (47), (here onwards, the distinction between square or row, or column matrices is omitted, but implied), one obtains,

$$\int_{V_n} B(\Delta\sigma^{\lambda\mu}, \sigma^{o\lambda\mu})\,\mathrm{d}V = \frac{1}{2}\int_{V_n} \Delta\sigma^T E\,\Delta\sigma\,\mathrm{d}V = \frac{1}{2}\int_{v} \{\beta^T(H^T E H)\beta + \Delta q^T(A^T E A)\,\Delta q$$
$$+ 2\beta^T(H^T E A)\,\Delta q + 2\beta^T(H^T E)\,\Delta\sigma_p + 2\Delta q^T(A^T E)\,\Delta\sigma_p + \text{constant}\}\,\mathrm{d}V \tag{48}$$

where

β = unknown stress coefficients

Δq = unknown generalized displacement increments

A^T = transpose of A, etc.

Carrying out the integration in (48) one obtains

$$\int_{V_n} B(\Delta\sigma^{\lambda\mu}, \sigma^{o\lambda\mu})\,\mathrm{d}V \equiv \tfrac{1}{2}\beta^T B\beta + \tfrac{1}{2}\,\Delta q^T C\,\Delta q + \beta^T D\,\Delta q + \beta^T\,\Delta Q_1 + \Delta q^T\,\Delta Q_2 \tag{49}$$

with obvious definitions for the matrices B, C, D, ΔQ and ΔQ_2.

Likewise,

$$\int_{\partial V_n} \Delta T^\lambda\,\Delta u_{\lambda L}\,\mathrm{d}S = \int_{\partial V_n} \Delta T^T\,\Delta u_L\,\mathrm{d}S = \int_{\partial V_n} [\beta^T(R^T L_B)\,\Delta q$$
$$+ (\Delta T_P^T L_B)\,\Delta q + \Delta q^T M^T L_B\,\Delta q)\,\mathrm{d}S \tag{50}$$

where L_B is the matrix for the boundary displacements given in equation (42).

Carrying out the integration in (50) one obtains

$$\int_{\partial V_n} \Delta T^\lambda\,\Delta u_\lambda\,\mathrm{d}S = \beta^T S\,\Delta q + T^T\,\Delta q + \Delta q^T P\,\Delta q \tag{51}$$

with obvious definition for the matrices S, T and P. Likewise, in the integral

$$\varepsilon^* = \int_V (-\sigma^{o\lambda\mu}\,\Delta u_{\lambda,\mu} + F^{o\lambda}\,\Delta u_\lambda)\,\mathrm{d}V + \int_{S_1} \bar{T}^{o\lambda}\,\Delta u_\lambda\,\mathrm{d}S. \tag{52}$$

Since $\sigma^{o\lambda\mu}$, $F^{o\lambda}$, $\bar{T}^{o\lambda}$ are known, the integral in equation (52), can in general be written

$$\varepsilon^* = [Q_c]\{\Delta q\}. \tag{53}$$

† Note the incremental form of the nonlinear constitutive law.

Thus, substituting equations (49, 51 and 53) into (38),

$$\pi_c^* = \sum_{n=1}^{N} \{-\tfrac{1}{2}\beta^T B\beta + \beta^T(S-D)\,\Delta q - \beta^T\,\Delta Q_1 - \tfrac{1}{2}\,\Delta q^T(C-2P)\,\Delta q$$
$$-\Delta q^T(\Delta Q_2 - T)\}_n - [Q_c]\{\Delta q\}. \tag{54}$$

We note that in equation (52) only the unknown stress coefficients β's are independent for each finite element. Thus, taking the variation with respect to β in each element, one obtains

$$-B\beta - \Delta Q_1 + (S-D)\,\Delta q = 0 \tag{55}$$

from which

$$\beta = -B^{-1}\,\Delta Q_1 + B^{-1}(S-D)\,\Delta q. \tag{56}$$

The substitution of equation (56) into equation (54) yields

$$\pi_c^* = \sum_{n=1}^{N} \{\tfrac{1}{2}\,\Delta q^T[(S-D)^T B^{-1}(S-D) - C + 2P]\,\Delta q - 2\,\Delta q^T(S-D)^T B^{-1}\,\Delta Q_1$$
$$-\Delta q^T(\Delta Q_2 - T) - [Q_c]\{\Delta q\}\}_n. \tag{57}$$

Taking the variation with respect to Δq,

$$\sum_{n=1}^{N} \{k\,\Delta q - Q\}_n = 0 \tag{58}$$

with

$$k = (S-D)^T B^{-1}(S-D) - C + (P+P^T) \tag{59}$$

$$\Delta Q = 2(S-D)^T B^{-1}\,\Delta Q_1 - \Delta Q_2 + T - Q_c \tag{60}$$

where k is the element stiffness matrix and ΔQ is the load matrix. We note that the vector Q_c of element generalized nodal forces (the last term on the right hand side of equation (60) which results from ε^* in equation (53) can be referred to as "residuals", since that they can be used to measure the residual error in nodal point equilibrium at the beginning of any load step.

If in equation (60), we set $C = 0$; $D = 0$; $\Delta Q_2 = 0$ and $P = 0$, then we recover the usual linear theory. Accordingly, we may rewrite equation (60) as

$$\begin{aligned}
k &= k_1 + k_2 \\
k_1 &= S^T B^{-1} S \\
k_2 &= D^T B^{-1} D - (D^T B^{-1} S + S^T B^{-1} D) - C + P + P^T \\
\Delta Q_1^* &= 2S^T B^{-1}\,\Delta Q_1 + T \\
\Delta Q_2^* &= -2D^T B^{-1}\,\Delta Q_1 - \Delta Q_2
\end{aligned} \tag{62}$$

where

k_1 = conventional linear hybrid stiffness matrix

k_2 = incremental stiffness matrix

ΔQ_1 = conventional load matrix

ΔQ_2 = incremental load matrix

Q_c = residuals to check equilibrium in reference state.

It can be easily verified that the incremental stiffness matrix is symmetric and positive semi-definite just as the conventional linear hybrid stress stiffness matrix (Pian and Tong [11]). Also, for reasons already discussed in Ref. [11], in general, the sum of the number of β's (stress coefficients) for each element and the number of rigid body degrees of freedom for each element should be in excess of the number of q's (generalized nodal displacements) of the same element.

Once an element shape is selected and appropriate choices for the field variables in each element as outlined earlier are made, the integrals in equation (62) can be evaluated numerically to find the element stiffness matrix and nodal load matrix. Employing the usual techniques of finite element assemblage, one can now derive a system of linear incremental equations, governing the behavior during the nth load step, for the entire structure,

$$([K_1]+[K_2])_N\{\Delta R\}_N = \{\Delta F_1\}_N+\{\Delta F_2\}_N+\{F_c\}_N \tag{63}$$

where matrices $[K_1]$, $[K_2]$, $\{\Delta R\}$, $\{\Delta F_1\}$, $\{\Delta F_2\}$ and $\{F_c\}$ are obtained by assembling the respective individual element matrices k_1, k_2, Δq, ΔQ_1^*, $Q\Delta_2^*$ and Q_c.

The procedure for the solution of equation (63), with the equilibrium check and corrective cycling procedure, has already been discussed in detail by Hofmeister, Greenbaum and Evensen [16], and is not repeated here. It should also be pointed out that at some stage in the process of incrementation, for example in state C_M, the matrix $[K_1+K_2]_M$ might become singular and $\{\Delta R\}_M$ can no longer be unique. This, in structural stability problems, the values of $\bar{F}_M^{o\lambda}$ and $\bar{T}_M^{o\lambda}$ corresponding to the case when $[K_1+K_2]_M$ is singular, are the critical loads. Pian and Tong [18], who use an incremental formulation and a compatible displacement model, suggest a procedure for solving equations of type (63) beyond these critical loads. However, for the hybrid stress model discussed here, such a procedure, in general, is more involved, and a discussion of the same is not presented here.

At this point it is worth noting that the total stresses $\sigma^{o\lambda\mu}+\Delta\sigma^{\lambda\mu}$ from load step N become initial stresses for step $N+1$. For step N, the stresses $\sigma^{o\lambda\mu}+\Delta\sigma^{\lambda\mu}$ where treated as the Piola–Kirchoff stresses referred to the state C_N before the addition of the nth load step. Thus, for treating the incremental problem corresponding to the $(N+1)$ the load increment, these total stresses $(\sigma^{o\lambda\mu}+\Delta\sigma^{\lambda\mu})$ at the end of step N must be converted to Piola–Kirchoff stresses $S^{o\lambda\mu}$ referred to the state C_{N+1} before the addition of the $(N+1)$th load increment (or equivalently the state at the end of nth step). Thus, following Green and Adkins [22],

$$S^{o\lambda\mu} = (I_3)^{-\frac{1}{2}}(\sigma^{o\lambda\mu}+\Delta\sigma^{\lambda\mu}) \tag{64}$$

where

$$I_3 = \det[\delta_\mu^\lambda+2\,\Delta e_\mu^\lambda]; \qquad \Delta e_\mu^\lambda = g^{\lambda\nu}\,\Delta e_{\mu\nu}. \tag{65}$$

In equations (65) $\Delta e_{\mu\nu}$ is the incremental Green's strain tensor in the nth load step as defined in equation (6).

Also, it is worth noting that when the volume and surface integrals for each discrete element considered in equations (49, 51 and 52) are evaluated for the element in the configuration C_N, the infinitesimal volume in state C_N is given by

$$dV = \sqrt{(g)}\, d\xi^1\, d\xi^2\, d\xi^3 \quad \text{in } C_N \tag{66}$$

where $g = \det|g_{\alpha\beta}|$ and $g_{\alpha\beta}$ is the metric in C_N. Due to incremental deformation in the nth load step, the discrete element shape and volume in C_{N+1} would be different from those in C_N. In C_{N+1}, the infinitesimal volume is given by

$$dV_1 = \sqrt{(G)}\, d\xi^1\, d\xi^2\, d\xi^3 \quad \text{in } C_{N+1} \tag{67}$$

where

$$G = \det|G_{\alpha\beta}| = \det|g_{\alpha\beta}| \cdot \det[\delta^{\lambda}_{\mu} + 2g^{\lambda\nu}\, \Delta e_{\mu\nu}] \tag{68}$$

where $G_{\alpha\beta}$ is the metric in C_{N+1} and $\Delta e_{\mu\nu}$ is given by equation (6). Thus, in evaluating the volume integrals for the deformed element in C_{N+1}, equation (67) is used for infinitesimal volume instead of (66), and the limits of integration on ξ^i are the same as those in C_N. Similar results can be derived for surface integrals.

AN EXAMPLE AND DISCUSSION

We consider, as an example, the large deflection of a shallow curved beam of uniform cross section with hinged but fixed ends subjected to laterally distributed load $p(x) = p_0 f(x)$ as shown in Fig. 1. It is assumed that the strains remain small and the stress–strain relations are linear. This problem has been solved analytically by Fung and Kaplan [24]. In the problem under consideration, the shallow beam has an initial shape (in the stress-free state)

$$y : C_1 \sin\frac{\pi x}{l} + C_2 \sin\frac{2\pi x}{l} \tag{69}$$

and is subjected to a half-sine shaped load distribution. If A is the area of cross-section and I the cross-section moment of inertia, the analytical solution [Ref. 24] has shown that for $C_1/2\sqrt{(A/I)} < \sqrt{(5{\cdot}5)}$ and $C_2 = 0$ the buckling of the curved beam will be of a

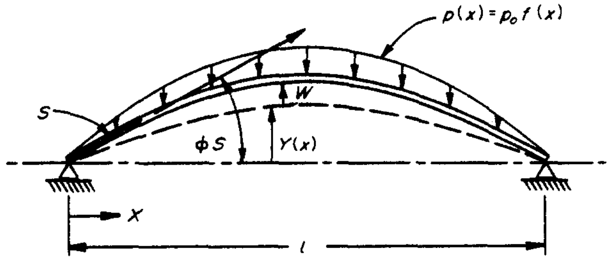

FIG. 1.

limit-load type and the deflection mode always follows the half-sine wave type. For $C_1/2\sqrt{(A/I)} > \sqrt{(5{\cdot}5)}$ and $C_2 = 0$ there exists, at sufficiently high loading, two branches in the exact solution, one symmetric and the other asymmetric. The latter corresponds to a bifurcation buckling mode. When the initial shape of beam is such that $C_2 \neq 0$, the pre-buckling deformation contains both symmetric and asymmetric components and the buckling phenomena is a limit load type.

In the example problem, using the analysis presented earlier, the length s along the beam in configuration C_1 (before the addition of first load-increment) is taken as the convected curvilinear coordinate. Considering configuration C_N, if ϕ is the angle between the base vector **g** and x-axis, it can be shown that the equilibrium equations for incremental Piola–Kirchoff stresses (per unit area in C_N), analogous to equation (39), become

$$\frac{\mathrm{d}\Delta N}{\mathrm{d}s}+\frac{\mathrm{d}\phi}{\mathrm{d}s}\frac{\mathrm{d}M}{\mathrm{d}s} = -\Delta p_s \tag{70}$$

and

$$\frac{\mathrm{d}^2\Delta M}{\mathrm{d}s^2}+\frac{\mathrm{d}\phi}{\mathrm{d}s}\Delta N = -\Delta P_n+\frac{\mathrm{d}}{\mathrm{d}s}\left[N_0\left(\frac{\mathrm{d}\Delta w}{\mathrm{d}s}\right)\right] \tag{71}$$

where N_0 is the initial longitudinal stress resultant in C_N, p_s and p_n are the nth tangential and normal (to beam) load increments, and ΔN and ΔM are the incremental stress resultant and stress couple respectively in the nth stop. The functional π_c^* analogous to equation (38) then becomes,

$$\pi_c^* = \sum_n \left\{-\int_{l_e}\frac{(\Delta M)^2}{2EI}+\frac{(\Delta N)^2}{2EA}\,\mathrm{d}s+\left[\Delta N\,\Delta u+\Delta V\,\Delta w+\Delta M\left(\frac{\mathrm{d}\Delta w}{\mathrm{d}s}-\frac{\Delta u}{R}\right)\right]_{b_e} -\int_{l_e} N_0\left(\frac{\mathrm{d}\Delta u}{\mathrm{d}s}-\frac{\Delta w}{R}\right)+\Delta M_0\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}\Delta w}{\mathrm{d}s}-\frac{\Delta u}{R}\right)\mathrm{d}s\right\} \tag{72}$$

where n is the number of elements, l_e the length of the element, and b_e refers to the two ends of the beam, and ΔV is the incremental transverse shear resultant at the ends of the element. In the numerical solution, three generalized displacements, Δu, Δw and $\mathrm{d}\Delta w/\mathrm{d}s$ are used at each node. The homogeneous solution (denoted by subscript H) corresponding to equations (70, 71) can be seen to be,

$$\begin{aligned}\Delta N_H &= \beta_1 \cos\phi+\beta_2 \sin\phi\\ \Delta M_H &= \beta_3+\int_0 (\beta_1 \sin-\beta_2 \cos\phi)\,\mathrm{d}s.\end{aligned} \tag{73}$$

It is observed that in more complicated problems such as plates and shells, recourse can be made to the so-called "static-geometric analogy" to find stress functions that identically satisfy the linear incremental homogeneous equations such as in equation (39). In order to obtain the particular solution, the given loads Δp_s and Δp_n are interpolated trigonometrically from their respective values at element nodes; and Δw in the element is interpolated by a four-parameter trigonometric function using the values Δw and $\mathrm{d}\Delta w/\mathrm{d}s$ at each node of the element. The trigonometric form of the particular solution, though cumbersome in form, can be obtained easily, in the form of equation (41), and is not recorded here.

The generation of the various metrics in equation (62) has been carried out numerically. In the example cases, discussed below, the incremental solution procedure used was the same as that discussed by Hofmeister *et al.* [16]; however, no corrective iterations were employed with the residual force vector F_{CN} (see equation 63), since the problem is essentially of small-strain. The incremental procedure is carried to that point in loading when the determinant of the total stiffness matrix at the current step has a different sign from that of the previous step.

Figure 2 shows plots of the load magnitude vs. the center deflection for the case $C_1/2\sqrt{(A/I)} = 1{\cdot}5$ and $C_2 = 0$ obtained the above method. For this solution, six elements

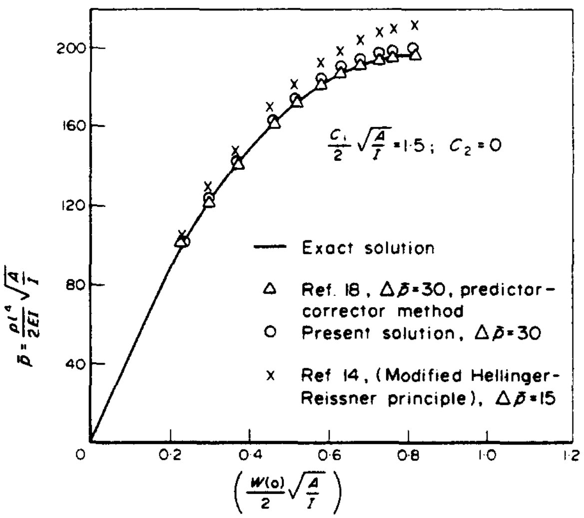

FIG. 2.

were used over the length of the beam. Also shown on the same figure are the results obtained by Pian and Tong [18] who use an incremental approach and the potential energy principle, and those by Pirotin [14] who uses a modified Hellinger–Reissner principle, as well as the analytical solution of Fung and Kaplan [24]. Both Refs. [18 and 14] are six elements over the length of the beam. Also, it should be pointed out that Pian and Tong [18], for the present problem with proportional loading characterized by the load parameter p_1, treat the incremental equations obtained by the potential energy principle as ordinary differential equations with the load parameter p_0 as the independent variable; and use a predictor–corrector method for their solution. It is found that the results obtained from the present method are comparable to these in [18] as to their accuracy. Figure 3 shows the variation of the center deflection with respect to load magnitude for the case $C/1\sqrt{(A/I)} = 3$ and $C_2/2\sqrt{(A/I)} = 0{\cdot}01$, using ten elements over the length of the beam. Again, the present results were found to be comparable to those in Ref. [17]. Further results from the application of the present method to the large-displacement analysis of an inflated toroidal shell will be presented later [25].

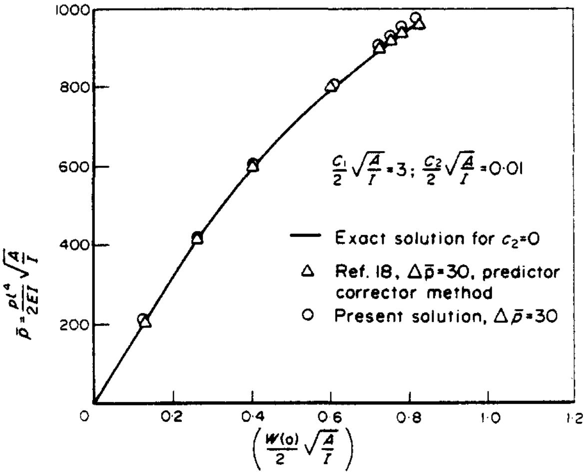

FIG. 3.

CONCLUSIONS

A consistent variational formulation of the hybrid stress finite element model for analysis of large deflection problems, in conjunction with an incremental approach, is presented. The concept of initial stresses is imployed. In each incremental step, the assumed incremental Piola–Kirchoff stresses satisfy the linearized incremental equilibrium equations in the interior of each finite element; whereas, a compatible incremental displacement field is assumed at the boundary of each element. A check on the equilibrium of initial stresses in the current reference geometry, before the addition of a further load-increment, is included to improve the numerical accuracy. The method leads to an incremental stiffness matrix and is easily adaptable to existing computer programs using the stiffness approach.

Since the hybrid stress model has proven to be a valuable tool in the linear analysis of complex problems such as sandwich plates and shells, and problems with singularities [Refs. 14 and 11], it can be expected to be of equal value in large deflection problems. Present results for the large deflections of a shallow beam show good agreement with existing results using displacement models. Further results on the application of the present method to the analysis of a toroidal shell will be presented later.

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Абстракт—Дается обобщение модели конечного элемента со смешанными напряжениями для задачи с большими прогибами. Эта модель первоначально предложена Пианом для линейных задач, в рамках малых перемещений. Используются разностный подход и понятия начальных напряжений. Приводится процесс проверки равновесия для исходного состояния, раньше добавления добавочного приращения нагрузки. Даются пример и обсуждение результатов.